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2005 J. Phys. A: Math. Gen. 38 L103

(http://iopscience.iop.org/0305-4470/38/7/L01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) L103–L114 doi:10.1088/0305-4470/38/7/L01

**LETTER TO THE EDITOR**

# Semiclassically concentrated solutions for the one-dimensional Fokker–Planck equation with a nonlocal nonlinearity

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Received 9 November 2004
Published 2 February 2005
Online at stacks.iop.org/JPhysA/38/L103

**Abstract**
Based on Maslov's complex germ method, a semiclassical asymptotic in a class of semiclassically concentrated functions is constructed for the one-dimensional Fokker–Planck equation with a nonlocal nonlinearity. The Einstein–Ehrenfest system describing the dynamics of mean values of coordinates and centred momenta is formulated. A nonlinear transition density is constructed.

PACS numbers: 03.65.Sq, 05.20−y, 05.70

## 1. Introduction

The Fokker–Planck equation[3] describes many interesting phenomena in thermodynamics, chemistry, evolutionary biology and quantitative social science where fluctuations are involved (see, e.g., [1, 2]). This equation has also been useful for studying the dynamical behaviour of stochastic differential equations driven by Gaussian noise. Recently there has been a revival of interest in describing a wide class of stochastic phenomena on the basis of the Fokker–Planck equation with different kinds of nonlinearities [3–20].

The concept of anomalous diffusion appears when the external stochastic indignation cannot be counted to be of Gaussian type. In this case a nonlinear Fokker–Planck equation with local nonlinearity [10] must be used. To mention only a few examples, maximum entropy solutions to nonlinear Fokker–Planck equations within a general thermostatistical formalism were constructed in [18]. Nonlinear Fokker–Planck equations admit peculiar time-dependent

[3] In the mathematical literature this equation is also called the forward Kolmogorov equation or the Fokker–Planck–Kolmogorov equation.

solutions of Tsallis MaxEnt form that are of relevance in describing the transport of fluids in porous media. Besides, nonlinear Fokker–Planck equations and their Tsallis MaxEnt solutions have been applied to the study of the process of correlated anomalous diffusion.

In [4, 20] different types of nonlinear Fokker–Planck equations with local and nonlocal nonlinearity were studied. Such nonlinear Fokker–Planck equations were shown to be related to the Renyi entropy, as well as to entropies proposed by Sharma and Mittal. The power-low-type equilibrium distributions of Tsallis thermostatistics were investigated from the viewpoint of nonequilibrium free energies, and the stability analysis of their solutions was carried out.

Methods of exact integration of equations of this class are few in number. By now, only a few particular cases of solving such equations are known, which are of considerable interest in view of the complexity of the problem. Worth mentioning in this context is the case presented in [20], where exact time-dependent solutions representing generalized one-dimensional Ornstein–Uhlenbeck processes were derived from nonlinear Fokker–Planck equations.

However, in general one is limited to the construction of stationary solutions. Then asymptotic methods of integration of such equations must be used. The WKB–Maslov method [21, 22] is one of the most advanced asymptotic methods. This work is devoted to applications of this method to the construction of asymptotic solutions of the nonlinear Fokker–Planck equation.

The Cauchy problem with a narrow initial distribution (i.e. with a small variance in the initial moment of time) for the Fokker–Planck equation is similar to that of constructing wave packets in quantum mechanics. Based on the complex WKB method (or Maslov's complex germ method) [22, 23] a procedure for building asymptotic solutions of the Schrödinger equation in the form of localized wave packets has been developed [24]. Solutions of such a kind were called semiclassically concentrated solutions (or states) which generalize the well-known coherent and squeezed states of quantum mechanics (see, e.g., [25, 26]). Another important application of this method is the construction of asymptotic solutions for the Cauchy problem related to the Hartree-type equation with a smooth potential [27, 28].

The purpose of this paper is to apply the asymptotic method to nonlinear Fokker–Planck equations (NLFPE) with a nonlocal nonlinearity for the case of narrow initial distributions

$$-D\frac{\partial u(x,t)}{\partial t} + \hat{\mathcal{H}}_\varkappa(t)u = 0, \tag{1.1}$$

$$\hat{\mathcal{H}}_\varkappa(t)u = D^2\frac{\partial}{\partial x}B(x,t)\frac{\partial u(x,t)}{\partial x} + D\frac{\partial}{\partial x}A(u,x,t)u(x,t), \tag{1.2}$$

$$A(u,x,t) = V_x(x,t) + \varkappa\int_{-\infty}^{\infty} W_x(x,y,t)u(y,t)\,\mathrm{d}y. \tag{1.3}$$

Here $u(x,t)$ is a normalized time-dependent probability distribution function, $D$ is a small parameter, $B(x,t)$, $V(x,t)$ and $W(x,y,t)$ are some fixed infinitesimal smooth functions growing at $|x|, |y| \to \infty$ not faster than a polynomial. In the equation above we used the notation $\phi_x = \partial\phi(x,y,t)/\partial x$. The function $B(x,t)$ can be treated as the diffusion coefficient, while $A(u,x,t)$ is the drift coefficient. Equation (1.1) is nonlinear because the drift coefficient (1.3) depends on the distribution function.

For a many-body system of identical particles the function $W(x,y,t)$ is the pair interaction potential. In this case (see, e.g., [5, 12, 19]) the original many-body problem has been reduced to the one-body problem with mean-field-type interaction. As a result, equation (1.1) involves a term representing the nonlocal feedback from the entire system. The nonlocal potential in equation (1.1) generalizes those studied in earlier works [3, 4, 6–8, 16–19].

Note that for a special class of equations (1.2) semiclassically concentrated solutions were constructed in [29]. Another interesting application can be found in [30] where a semiclassical asymptotic concentrated on invariant Lagrange tori for the Fokker–Planck–Kolmogorov equation was constructed.

In this work we consider the Cauchy problem for the one-dimensional Fokker–Planck equation (1.1) and construct a semiclassical asymptotic (with a power accuracy $O(D^{3/2})$, $D \to 0$) in a class of semiclassically concentrated functions. The nonlinear evolution operator and the nonlinear transition density are obtained (with the same accuracy) in an explicit form in the functional space chosen. The key point of the method is the integration of an auxiliary system of ordinary differential equations (the Einstein–Ehrenfest system). Solutions of this system allow one to construct an associated linear Fokker–Planck equation (ALFPE). In turn, solutions of the latter make it possible to find those of the nonlinear Fokker–Planck equation with a given accuracy.

## 2. Trajectory-concentrated functions

A class of trajectory concentrated functions is the key point in the asymptotic integration method proposed in [24, 28]. Let us search for an asymptotic solution to equation (1.1) in a class of functions depending on the trajectory $x = X(t, D)$, a real function $S(t, D)$ (an analogue of the classical action functional) and a small parameter $D$. As $D \to 0$ these functions are concentrated in a neighbourhood of a point moving along a given curve in the configuration space $x = X(t, 0)$ and are called trajectory-concentrated functions.

In what follows we denote the class of such functions as $\mathfrak{P}_t^D(X(t, D), S(t, D))$ and specify it in the following way:

$$\begin{aligned}\mathfrak{P}_t^D &= \mathfrak{P}_t^D(X(t, D), S(t, D)) \\ &= \left\{\Phi : \Phi(x, t, D) = \varphi\left(\frac{\Delta x}{\sqrt{D}}, t, D\right)\exp\left[\frac{1}{D}S(t, D)\right]\right\},\end{aligned} \tag{2.1}$$

where the real function $\varphi(\eta, t, D)$ belongs to the Schwartz space $\mathbb{S}$ in the variable $\eta \in \mathbb{R}$. Besides, the function is assumed to have a smooth dependence on $t$ and a regular dependence on $\sqrt{D}$, as $D \to 0$. In the equation above we denoted $\Delta x = x - X(t, D)$ and the real functions $S(t, D)$ $X(t, D)$ characterizing the class $\mathfrak{P}_t^D(X(t, D), S(t, D))$ depend on $\sqrt{D}$ regularly in a neighbourhood of $D = 0$. In what follows, where it does not cause a misunderstanding, we use a shorthand $\mathfrak{P}_t^D$ for $\mathfrak{P}_t^D(X(t, D), S(t, D))$.

As was shown in [24, 29], for a function from the class $\mathfrak{P}_t^D$ the asymptotic estimations for

$$\begin{aligned}&\frac{\|\hat{p}^k\Delta x^l u\|}{\|u\|} = O(D^{(k+l)/2}), \qquad D \to 0, \\ &\frac{\|[D\partial_t + \dot{X}(t, D)D\partial_x - \dot{S}(t, D)]u\|}{\|u\|} = O(D), \qquad D \to 0,\end{aligned} \tag{2.2}$$

hold, where $\hat{p} = D\partial_x$. In particular, from equation (2.2) one finds the following estimations for the operators $\hat{p} = \widehat{O}(\sqrt{D})$, $\Delta x = \widehat{O}(\sqrt{D})$.

In constructing asymptotic solutions, it is useful to define, along with the class $\mathfrak{P}_t^D$, the following class of functions:

$$\begin{aligned}\mathfrak{C}_t^D &= \mathfrak{C}_t^D(X(t, D), S(t, D)) \\ &= \left\{\Phi : \Phi(x, t, D) = \varphi\left(\frac{\Delta x}{\sqrt{D}}, t\right)\exp\left[\frac{1}{D}S(t, D)\right]\right\},\end{aligned} \tag{2.3}$$

where the functions $\varphi$ are independent of $D$. It is obvious that $\mathfrak{C}_t^D \in \mathfrak{P}_t^D$.

## 3. Einstein–Ehrenfest system

In [24] a new approach to the semiclassical approximation in nonrelativistic mechanics has been proposed. It was shown that, on the class of trajectory-concentrated functions, the semiclassical approximation is equivalent to the replacement of the Schrödinger equation by a finite system of ordinary differential equations of first order. In stochastic systems this approach is based on the following obvious fact: the description of a stochastic system in terms of its probability distribution function is completely equivalent to that in terms of an infinite system of centred moments. The latter are uniquely specified by the distribution function (moments problem) which, in turn, is completely determined by an infinite system of moments. In contrast to the linear Fokker–Planck equation, for the nonlinear equation (1.1) a set of equations for centred moments is essentially used for the semiclassically concentrated solutions.

Let us assume that for the nonlinear equation (1.1) there exist exact (or differing by a quantity of order $O(D^\infty)$) solutions in the class of trajectory-concentrated functions with the initial value $\varphi(x, D) \in \mathfrak{P}_0^D$. The mean values of the operators $x$ and $(\Delta x)^k$, $k = \overline{1, \infty}$ corresponding to the solution $u(x, t, D)$ will be denoted by

$$x_u(t, D) = \langle x\rangle = \int_{-\infty}^{\infty} x u(x, t, D)\,\mathrm{d}x, \tag{3.1}$$

$$\alpha_u^{(k)}(t, D) = \langle(\Delta x)^k\rangle = \int_{-\infty}^{\infty} [x - x_u(t, D)]^k u(x, t, D)\,\mathrm{d}x. \tag{3.2}$$

The mean value of the operator $x$ evolves according to the relation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}\rangle &= \int_{-\infty}^{\infty} x u_t(x, t, D)\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty} x\left\{D[B(x, t)u_x(x, t)]_x + \left[V_x(x, t)u(x, t)\right.\right.\\ &\quad \left.\left. +\varkappa u(x, t)\int_{-\infty}^{\infty} W_x(x, y, t)u(y, t)\,\mathrm{d}y\right]_x\right\}\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\left\{DB_x(x, t) - V_x(x, t) - \varkappa\int_{-\infty}^{\infty} W_x(x, y, t)u(y, t)\,\mathrm{d}y\right\}u(x, t)\,\mathrm{d}x.\end{aligned}$$

Then one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}\rangle = \left\langle\left[DB_x(x, t) - V_x(x, t) - \varkappa\int_{-\infty}^{\infty} W_x(x, y, t)u(y, t)\,\mathrm{d}y\right]\right\rangle. \tag{3.3}$$

In turn, the mean value of the operator $(\Delta x)^k$, $k = \overline{1, \infty}$ is determined by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle(\Delta x)^n\rangle &= \int_{-\infty}^{\infty}(\Delta x)^n u_t(x, t, D)\,\mathrm{d}x - n\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t}\int_{-\infty}^{\infty}(\Delta x)^{n-1}u(x, t, D)\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty}(\Delta x)^n\left\{D[B(x, t)u_x(x, t)]_x + \left[V_x(x, t)u(x, t)\right.\right.\\ &\quad\left.\left. +\varkappa u(x, t)\int_{-\infty}^{\infty} W_x(x, y, t)u(y, t)\,\mathrm{d}y\right]_x\right\}\mathrm{d}x - n\dot{x}_u(t, D)\alpha_u^{(n-1)}(t, D)\\ &= -n\int_{-\infty}^{\infty}(\Delta x)^{n-1}\left\{DB_x(x, t) - V_x(x, t)\right.\end{aligned}$$

$$
- \varkappa \int_{-\infty}^{\infty} W_x(x,y,t) u(y,t)\,\mathrm{d}y \Big\} u(x,t)\,\mathrm{d}x
$$
$$
+ Dn(n-1) \int_{-\infty}^{\infty} (\Delta x)^{n-2} B(x,t) u(x,t)\,\mathrm{d}x - n\dot{x}_u(t,D)\alpha_u^{(n-1)}(t,D),
$$

which gives

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle(\Delta x)^n\rangle = -n\left\langle(\Delta x)^{n-1}\left[DB_x(x,t) - V_x(x,t) - \varkappa\int_{-\infty}^{\infty} W_x(x,y,t)u(y,t)\,\mathrm{d}y\right]\right\rangle
$$
$$
+ Dn(n-1)\langle(\Delta x)^{n-2}B(x,t)\rangle - n\dot{x}_u(t,D)\alpha_u^{(n-1)}(t,D). \tag{3.4}
$$

In particular, by making use of estimation (2.2) for the mean values of the operators $x$ and $(\Delta x)^k$, $k = \overline{1, M}$ one has

$$
\dot{x}_u = -\sum_{k=0}^{M}\left\{\frac{1}{k!}\left[\frac{\partial^k V_x(t)}{\partial x^k} - D\frac{\partial^k B_x(t)}{\partial x^k}\right]\alpha_u^{(k)} + \varkappa\sum_{l=0}^{M}\frac{1}{k!l!}\frac{\partial^{k+l}W_x(t)}{\partial x^k\partial y^l}\alpha_u^{(k)}\alpha_u^{(l)}\right\} + O(D^{(M+1)/2}), \tag{3.5}
$$

$$
\dot{\alpha}_u^{(n)} = -n\sum_{k=0}^{M-n+1}\left\{\frac{1}{k!}\left[\frac{\partial^k V_x(t)}{\partial x^k} - D\frac{\partial^k B_x(t)}{\partial x^k}\right]\alpha_u^{(k+n-1)} + \varkappa\sum_{l=0}^{M}\frac{1}{k!l!}\frac{\partial^{k+l}W_x(t)}{\partial x^k\partial y^l}\alpha_u^{(k+n-1)}\alpha_u^{(l)}\right\}
$$
$$
+ Dn(n-1)\sum_{k=0}^{M-n+1}\frac{1}{k!}\frac{\partial^k B(t)}{\partial x^k}\alpha_u^{(k+n-2)} - n\dot{x}_u\alpha_u^{(n-1)} + O(D^{(M+1)/2}). \tag{3.6}
$$

Let us consider then a system of ordinary differential equations with respect to the variables $x$ and $\alpha^{(n)}$, $n = \overline{1, M}$,

$$
\begin{cases}
\dot{x} = -\displaystyle\sum_{k=0}^{M}\left\{\frac{1}{k!}\left[\frac{\partial^k V_x(t)}{\partial x^k} - D\frac{\partial^k B_x(t)}{\partial x^k}\right]\alpha^{(k)} + \varkappa\sum_{l=0}^{M}\frac{1}{k!l!}\frac{\partial^{k+l}W_x(t)}{\partial x^k\partial y^l}\alpha^{(k)}\alpha^{(l)}\right\}, \\
\dot{\alpha}^{(n)} = -n\displaystyle\sum_{k=0}^{M-n+1}\left\{\frac{1}{k!}\left[\frac{\partial^k V_x(t)}{\partial x^k} - D\frac{\partial^k B_x(t)}{\partial x^k}\right]\alpha^{(k+n-1)}\right. \\
\quad \left. + \varkappa\displaystyle\sum_{l=0}^{M}\frac{1}{k!l!}\frac{\partial^{k+l}W_x(t)}{\partial x^k\partial y^l}\alpha^{(k+n-1)}\alpha^{(l)}\right\} + Dn(n-1)\sum_{k=0}^{M-n+1}\frac{1}{k!}\frac{\partial^k B(t)}{\partial x^k}\alpha^{(k+n-2)} \\
\quad + n\alpha^{(n-1)}\displaystyle\sum_{k=0}^{M}\left\{\frac{1}{k!}\left[\frac{\partial^k V_x(t)}{\partial x^k} - D\frac{\partial^k B_x(t)}{\partial x^k}\right]\alpha^{(k)} + \varkappa\sum_{l=0}^{M}\frac{1}{k!l!}\frac{\partial^{k+l}W_x(t)}{\partial x^k\partial y^l}\alpha^{(k)}\alpha^{(l)}\right\}.
\end{cases} \tag{3.7}
$$

We define the initial conditions for system (3.7) according to the prescription

$$
\begin{aligned}
x|_{t=0} &= x_\varphi = \int_{-\infty}^{\infty} x\varphi(x,D)\,\mathrm{d}x, \\
\alpha^{(k)}|_{t=0} &= \alpha_\varphi^{(k)} = \int_{-\infty}^{\infty}(x - x_\varphi)^k\varphi(x,D)\,\mathrm{d}x, \qquad k = \overline{1, M}.
\end{aligned} \tag{3.8}
$$

In what follows we call the system of equations (3.8) *the Einstein–Ehrenfest system of order M* for the NLFPE. The solutions of the system will be denoted by

$$
x = x_\varphi^{(M)}(t,D), \qquad \alpha^{(k)} = \alpha_\varphi^{(k,M)}(t,D), \qquad k = \overline{1, M}, \tag{3.9}
$$

$$
y_\varphi^{(M)}(t,D) = \big(x_\varphi^{(M)}(t,D), \alpha_\varphi^{(1,M)}(t,D), \alpha_\varphi^{(2,M)}(t,D), \ldots, \alpha_\varphi^{(M,M)}(t,D)\big). \tag{3.10}
$$

The following statement holds true.

**Proposition 1.** *The solutions of the Einstein–Ehrenfest system of order $M$, $x_\varphi^{(M)}(t, D)$ and $\alpha_\varphi^{(k,M)}(t, D)$, $k = \overline{1, M}$, and the mean values (3.1), (3.2) are connected according to the rule*

$$\begin{aligned} x_u(t, D) &= x_\varphi^{(M)}(t, D) + O(D^{(M+1)/2}), && t \in [0, T] \\ \alpha_u^{(k)}(t, D) &= \alpha_\varphi^{(k,M)}(t, D) + O(D^{(M+1)/2}), && k = \overline{1, M}, \end{aligned} \tag{3.11}$$

*where $u = u(x, t, D)$ is a semiclassically concentrated solution of the NLFPE with the initial condition $\varphi(x, D) \in \mathfrak{P}_0^D$.*

It can readily be noted that the solutions of the Einstein–Ehrenfest system may be useful for the Fokker–Planck equation with a local nonlinearity. Thus, the condition of the central moment of the second order will be limited in $t$, i.e. $|\alpha^{(2)}| \leqslant M$, $M = \text{const}$. For the nonlinear Schrödinger equation it corresponds to the condition of the existence (nonexistence) of solitons [28].

## 4. Associated LFPE

We will search for a solution to equation (1.1) in the class $\mathfrak{P}_t^D(x_u(t, D), S(t, D))$ (2.1), with the function $S(t, D)$ to be specified below. To this end, let us Taylor expand the function $W(x, y, t)$ in powers of $\Delta y = y - x_u(t, D)$

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial}{\partial x}B(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial}{\partial x}\left[\left(V_x(x,t) + \varkappa\sum_{l=0}^{M}\frac{1}{l!}\frac{\partial^l W_x(x,y,t)}{\partial y^l}\bigg|_{y=x_u(t,D)}\alpha_u^{(l)}(t,D)\right)u(x,t)\right] + O(D^{(M+1)/2}). \tag{4.1}$$

Let us consider an auxiliary equation which follows from equation (4.1) after the change of the mean values (3.2) by the solutions of the Einstein–Ehrenfest system of order $M$ (3.7)

$$\frac{\partial v(x,t)}{\partial t} = D\frac{\partial}{\partial x}B(x,t)\frac{\partial v(x,t)}{\partial x} + \frac{\partial}{\partial x}[\Lambda^{(M)}(x,t)v(x,t)] + O(D^{(M+1)/2}), \tag{4.2}$$

$$\Lambda^{(M)}\big(x,t,y_\varphi^{(M)}(t,D)\big) = V_x(x,t) + \varkappa\sum_{l=0}^{M}\frac{1}{l!}\frac{\partial^l W_x\big(x, x_\varphi^{(M)}(t,D), t\big)}{\partial y^l}\alpha_\varphi^{(l,M)}(t,D). \tag{4.3}$$

Making use of estimations (3.11), one arrives at

**Proposition 2.** *If semiclassically concentrated solutions of equations (4.1) and (4.2) obey one and the same initial condition*

$$u|_{t=0} = v|_{t=0} = \varphi(x, D), \qquad \varphi(x, D) \in \mathfrak{P}_0^D, \tag{4.4}$$

*then the solutions of (4.1) and (4.2) are connected in the following way:*

$$u^{(M)}(x, t, D) = v^{(M)}(x, t, D) + O(D^{(M+1)/2}), \tag{4.5}$$

*where $u(x, t, D)$ is a semiclassically concentrated solution of the NLFPE with the initial condition (4.4).*

In what follows we call equation (4.2) the associated linear Fokker–Planck equation of order $M$.

Taking into account the estimations (2.2), which are valid in the class $\mathfrak{P}_t^D\big(x_\varphi^{(M)}(t, D), S(t, D)\big)$, let us represent equation (4.2) in the form

$$\big[\hat{L}_0^{(M)}\big(y_\varphi^{(M)}(t,D)\big) + \sqrt{D}\hat{L}_1^{(M)}\big(y_\varphi^{(M)}(t,D)\big) + D\hat{L}_2^{(M)}\big(y_\varphi^{(M)}(t,D)\big) + \cdots\big]v = 0, \tag{4.6}$$

where

$$\begin{aligned}
\hat{L}_0^{(M)}\big(y_\varphi^{(M)}(t,D)\big) &= D\partial_t + \dot{x}_\varphi^{(M)}(t,D)D\partial_x \\
&\quad - \Lambda_x^{(M)}\big(x_\varphi^{(M)}(t,D),t\big)D\partial_x\Delta x - B\big(x_\varphi^{(M)}(t,D),t\big)D^2\partial_x^2 \\
\sqrt{D}\hat{L}_1^{(M)}\big(y_\varphi^{(M)}(t,D)\big) &= -\frac{1}{2}\Lambda_{xx}^{(M)}\big(x_\varphi^{(M)}(t,D),t\big)D\partial_x\Delta x^2 - B_x\big(x_\varphi^{(M)}(t,D),t\big)D\partial_x\Delta x D\partial_x \\
&\quad + \left\{\frac{1}{1!}\left[\frac{\partial V_x(t)}{\partial x} - D\frac{\partial B_x(t)}{\partial x}\right]\alpha^{(1)}(t,D)\right. \\
&\quad \left. + \varkappa\sum_{k+l=1}\frac{1}{k!l!}\frac{\partial^{k+l}W_x(t)}{\partial x^k\partial y^l}\alpha^{(k)}(t,D)\alpha^{(l)}(t,D)\right\}D\partial_x, \\
D\hat{L}_2^{(M)}\big(y_\varphi^{(M)}(t,D)\big) &= -\frac{1}{3!}\Lambda_{xxx}^{(M)}\big(x_\varphi^{(M)}(t,D),t\big)D\partial_x\Delta x^3 \\
&\quad - \frac{1}{2}B_{xx}\big(x_\varphi^{(M)}(t,D),t\big)D\partial_x(\Delta x)^2D\partial_x \\
&\quad + \left\{\frac{1}{2!}\left[\frac{\partial^2 V_x(t)}{\partial x^2} - D\frac{\partial^2 B_x(t)}{\partial x^2}\right]\alpha^{(2)}(t,D)\right. \\
&\quad \left. + \varkappa\sum_{k+l=2}\frac{1}{k!l!}\frac{\partial^{k+l}W_x(t)}{\partial x^k\partial y^l}\alpha^{(k)}(t,D)\alpha^{(l)}(t,D)\right\}D\partial_x, \\
&\quad \ldots.
\end{aligned} \tag{4.7}$$

It follows from equation (2.2) that $\sqrt{D^k}\hat{L}_k\big(y_\varphi^{(M)}(t,D)\big) = \hat{O}(D^{k/2})$.

Let us search for the trajectory-concentrated solutions of equation (4.6) $\big(u(x,t,D) \in \mathfrak{P}_t^D\big)$ in the form

$$v(x,t,D) = v^{(0)}(x,t,D) + \sqrt{D}v^{(1)}(x,t,D) + Dv^{(2)}(x,t,D) + \cdots, \tag{4.8}$$

where $v^{(k)}(x,t,D) \in \mathfrak{C}_t^D\big(x_\varphi^{(M)}(t,D), S(t,D)\big)$. Substituting (4.8) in (4.6) and comparing the terms having the same estimation in $D$ (in the sense of (2.2)) one arrives at the recurrent system of equations

$$\begin{aligned}
&\hat{L}_0^{(M)}\big(y_\varphi^{(M)}(t,D)\big)v^{(0)} = 0, \\
&\hat{L}_0^{(M)}\big(y_\varphi^{(M)}(t,D)\big)v^{(1)} + \hat{L}_1^{(M)}\big(y_\varphi^{(M)}(t,D)\big)v^{(0)} = 0, \\
&\hat{L}_0^{(M)}\big(y_\varphi^{(M)}(t,D)\big)v^{(2)} + \hat{L}_1^{(M)}\big(y_\varphi^{(M)}(t,D)\big)v^{(1)} + \hat{L}_2^{(M)}\big(y_\varphi^{(M)}(t,D)\big)v^{(0)} = 0, \\
&\ldots.
\end{aligned} \tag{4.9}$$

We then look for a particular solution to equation (4.9) and take the ansatz

$$v_0^{(0)}(x,t,D) = N_D\exp\left\{\frac{1}{D}\left[S(t,D) + \frac{1}{2}Q(t)\Delta x^2\right]\right\}, \tag{4.10}$$

where $N_D$ is a normalization constant and $Q(t)$ will be fixed below. Plugging equation (4.10) into equation (4.9) and comparing terms of the same power in $\Delta x$, one ends up with the following system of equations:

$$\dot{S} - D\Lambda_x^{(M)}\big(x_\varphi^{(M)}(t,D),t\big) - 2DB\big(x_\varphi^{(M)}(t,D),t\big)Q = 0, \tag{4.11}$$

$$\tfrac{1}{2}\dot{Q} - \Lambda_x^{(M)}\big(x_\varphi^{(M)}(t,D),t\big)Q - 2B\big(x_\varphi^{(M)}(t,D),t\big)Q^2 = 0. \tag{4.12}$$

From equation (4.11) one finds that

$$S(t, D) = D\int_0^t \left[\Lambda_x^{(M)}\big(x_\varphi^{(M)}(t, D), t\big) + 2B\big(x_\varphi^{(M)}(t, D), t\big)Q(t)\right] \mathrm{d}t. \qquad (4.13)$$

In analogy with the Hartree-type equation [27, 28], we then search for a symmetry operator of equation (4.9) in the form

$$\hat{a}(t) = N_a[Z(t)\hat{p} - W(t)\Delta x], \qquad (4.14)$$

where $N_a$ is a constant. The functions $Z(t)$ and $W(t)$ will be fixed from the condition

$$\left[\hat{L}_0\big(y_\varphi^{(M)}(t, D)\big), \hat{a}(t)\right] = 0. \qquad (4.15)$$

Equation (4.15) holds if $Z(t)$ and $W(t)$ obey the system of equations

$$\begin{cases} \dot{W} = -\Lambda_x^{(M)}\big(x_\varphi^{(M)}(t, D), t\big)W, \\ \dot{Z} = 2B\big(x_\varphi^{(M)}(t, D), t\big)W + \Lambda_x^{(M)}\big(x_\varphi^{(M)}(t, D), t\big)Z. \end{cases} \qquad (4.16)$$

We will call (4.16) the system in variations for equation (4.5). One can prove the validity of the relations (the details can be found in [24])

$$Q(t) = \frac{W(t)}{Z(t)}, \qquad \exp\left[\frac{1}{D}S(t, D)\right] = \sqrt{\frac{Z(0)}{Z(t)}}, \qquad (4.17)$$

which also imply the equality

$$v_0^{(0)}(x, t, D) = N_D\sqrt{\frac{Z(0)}{Z(t)}}\exp\left[\frac{1}{2D}\frac{W(t)}{Z(t)}\Delta x^2\right]. \qquad (4.18)$$

For the system in variations (4.16) let us choose the initial conditions

$$W|_{t=0} = \pm b, \qquad Z|_{t=0} = 1, \qquad b > 0, \qquad (4.19)$$

and denote the solution of (4.16) as

$$a^{\pm}(t) = (W^{\pm}(t), Z^{\pm}(t)). \qquad (4.20)$$

The sign in the latter correlates with that in equation (4.19).

Because the skew-orthogonal scalar product is conserved on solutions of the system in variations (4.16)

$$\{a^-(t), a^+(t)\} = W^-(t)Z^+(t) - Z^-(t)W^+(t) = -2b, \qquad (4.21)$$

vectors (4.20) provide conjugate solutions.

Let us choose the constant $N_{a^{\pm}}$ in operators (4.14) assuming that the condition

$$[\hat{a}^-(t), \hat{a}^+(t)] = 1, \qquad (4.22)$$

holds. Then one finds

$$\hat{a}^{\pm}(t) = \frac{1}{\sqrt{2bD}}[Z^{\pm}(t)\hat{p} + W^{\pm}(t)\Delta x]. \qquad (4.23)$$

Note that the function

$$v_0^{(0)}(x, t, D) = \frac{N_D}{\sqrt{Z^-(t)}}\exp\left[\frac{1}{2D}\frac{W^-(t)}{Z^-(t)}\Delta x^2\right], \qquad N_D = \sqrt[4]{\frac{b}{D\pi}} \qquad (4.24)$$

is a ground state for the operator $\hat{a}^-(t)$:

$$\hat{a}^-(t)v_0^{(0)}(x, t, D) = 0. \qquad (4.25)$$

As a next step, we consider the set of functions having the form

$$v_n^{(0)}(x,t,D) = \frac{1}{\sqrt{n!}}(\hat{a}^+(t))^n v_0^{(0)}(x,t,D). \tag{4.26}$$

Making use of the Rodrigues formula for the Hermite polynomials [31]

$$H_n(\xi) = (-1)^n \, \mathrm{e}^{\xi^2}(\mathrm{e}^{-\xi^2})^{(n)},$$

one finds

$$v_n^{(0)}(x,t,D) = \frac{(-1)^n}{\sqrt{2^n n!}}\left[\frac{Z^-(t)}{Z^+(t)}\right]^{n/2} H_n\left(\sqrt{\frac{b}{DZ^+(t)Z^-(t)}}\Delta x\right) v_0^{(0)}(x,t,D). \tag{4.27}$$

As a consequence of the definitions of $\hat{a}^{\pm}(t)$ and $v_0^{(0)}(x,t,D)$, functions (4.27) solve equations (4.9) for an arbitrary natural $n$.

Now, let us consider a system of auxiliary functions

$$w_n^{(0)}(x,t,D) = \frac{(-1)^n}{\sqrt{2^n n!}}\left[\frac{Z^+(t)}{Z^-(t)}\right]^{n/2} H_n\left(\sqrt{\frac{b}{DZ^+(t)Z^-(t)}}\Delta x\right) w_0^{(0)}(x,t,D), \tag{4.28}$$

where

$$w_0^{(0)}(x,t,D) = \frac{N_D}{\sqrt{Z^+(t)}} \exp\left[\frac{1}{2D}\frac{W^+(t)}{Z^+(t)}\Delta x^2\right]. \tag{4.29}$$

One can prove that systems (4.27) and (4.28) are bi-orthonormal.

By analogy with the Schrödinger and Hartree-type equations, we will call functions (4.27) *semiclassical trajectory-coherent solutions* of the Fokker–Planck equation (2.1).

The following theorem holds:

**Theorem 1.** 1. *The functions $v_n^{(0)}(x,t,D)$ are asymptotic solutions of equation (2.1), to the order $O(D^{3/2})$ and obey the condition*

$$u|_{t=0} = \varphi(x,D) = v_n^{(0)}(x,0,D). \tag{4.30}$$

2. *The bi-orthogonal system of functions $\{v_n^{(0)}(x,t,D)\}_{n=0}^{\infty}$ (4.27), $\{w_n^{(0)}(x,t,D)\}_{n=0}^{\infty}$ (4.28) is complete in $L_2(\mathbb{R}_x^n)$.*

## 5. Transition density in a class of trajectory-concentrated functions

In this section we will construct a transition density $G^{(0)}(x,y,t)$ (a Green function, a kernel of the evolution operator) for equation (3.10)

$$\hat{L}_0\big(y_\varphi^{(M)}(t,D)\big)G^{(0)} = 0, \qquad G^{(0)}|_{t=0} = \delta(x-y). \tag{5.1}$$

Because the system (4.27), (4.28) is bi-orthonormal and the functions $u_n^{(0)}(x,t,D)$ solve equation (3.10) for any $n$, the transition density (5.1) can be represented in the form

$$G^{(0)}\big(x,y,t,s,D,y_\varphi^{(M)}(t,D)\big) = \sum_{n=0}^{\infty} v_n^{(0)}(x,t,D)w_n^{(0)}(y,s,D). \tag{5.2}$$

Substituting (4.27), (4.28) in (5.2) and taking into account the explicit form of $v_n^{(0)}(x,t,D)$ and $w_n^{(0)}(y,t,D)$ one obtains

$$G^{(0)}\big(x, y, t, s, D, y_\varphi^{(M)}(t, D)\big)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \sqrt{\frac{b}{D\pi}} \frac{1}{\sqrt{Z^-(t)Z^+(s)}} \left[\frac{Z^-(t)}{Z^+(t)} \frac{Z^+(s)}{Z^-(s)}\right]^{n/2}$$

$$\times H_n\left(\sqrt{\frac{b}{DZ^+(t)Z^-(t)}}\Delta x\right) H_n\left(\sqrt{\frac{b}{DZ^+(s)Z^-(s)}}\Delta y\right)$$

$$\times \exp\left\{\frac{1}{2D}\left[\frac{W^-(t)}{Z^-(t)}\Delta x^2 - \frac{W^+(s)}{Z^+(s)}\Delta y^2\right]\right\}, \tag{5.3}$$

where $\Delta y = y - x_0$. Exploiting then Meler's formula [31]

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2z}\right)^n H_n(x)H_n(y) = \frac{1}{\sqrt{1-z^2}} \exp\left[\frac{2xyz - z^2(x^2+y^2)}{1-z^2}\right], \tag{5.4}$$

for equation (5.3) one finds

$$G^{(0)}\big(x, y, t, s, D, y_\varphi^{(M)}(t, D)\big)$$

$$= \sqrt{\frac{b}{D\pi}} \frac{1}{\sqrt{Z^-(t)Z^+(s) - Z^-(s)Z^+(t)}} \exp\left\{\frac{1}{2D}\left[\frac{W^-(t)\Delta x^2}{Z^-(t)} + \frac{W^+(s)\Delta y^2}{Z^+(s)}\right]\right\}$$

$$\times \exp\left\{\frac{b}{D}\left[\frac{[2Z^-(t)Z^+(s)\Delta x\Delta y - Z^+(s)Z^-(s)\Delta x^2 - Z^+(t)Z^-(t)\Delta y^2]}{[Z^-(t)Z^+(s)][Z^-(t)Z^+(s) - Z^+(t)Z^-(s)]}\right]\right\}. \tag{5.5}$$

Let us denote an operator with the kernel $G^{(0)}\big(x, y, t, D, y_\varphi^{(M)}(t, D)\big)$ as $\hat{U}^{(0)}\big(t, y_\varphi^{(M)}(t, D)\big)$. Then for the function

$$u_\varphi^{(0)}(x, t, D) = \hat{U}^{(0)}\big(t, y_\varphi^{(M)}(t, D)\big)\varphi(x, D)$$

$$= \int_{-\infty}^{\infty} G^{(0)}\big(x, y, t, D, y_\varphi^{(M)}(t, D)\big)\varphi(y, D)\,\mathrm{d}y, \tag{5.6}$$

where $\varphi(x, D) \in \mathfrak{P}_0^D$, the following statement holds.

**Proposition 3.** *The function $u_\varphi^{(0)}(x, t, D)$ is an asymptotic solution of equation (1.1), to the order $O(D^{3/2})$ which obeys the initial condition*

$$u|_{t=0} = \varphi(x, D). \tag{5.7}$$

In concluding the paper we point out that the expressions constructed in this work for the transition density of the nonlinear Fokker–Planck equation (1.1) can be generalized to the case of both a nonlinear Fokker–Planck equation in a multidimensional space with smooth coefficients of a general form and fractional Fokker–Planck-like equations. This can be done using the results of [24, 27, 28]. This generalization will be valid in the sense of an approximation to within $\widehat{O}(D^{(M+1)/2})$, $D \to 0$, where $M$ is the order of the Einstein–Ehrenfest system. Besides, mathematical constructions developed in this work could be helpful for the description of a pairwise interaction in a system of particles subjected to regular and random forces.

## Acknowledgments

This work was partially supported by a NATO Collaborative Linkage Grant No PST.CLG.979389. The research of SB is supported in part by a European Community's

Human Potential Programme contract HPRN-CT-2000-00131, a INTAS grant no 00-0254 and an Italian Research Ministry MIUR, National Interest Program, under grant COFIN 2002022534. AT has been supported by the President of Russian Federation, Grant No MD-246.2003.02.

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